A bakery owner turns on his doughnut machine at 8:30 AM. At 11:10 AM the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

(A) 1:50 PM **(B)** 3:00 PM **(C)** 3:30 PM **(D)** 4:30 PM **(E)** 5:50 PM

2008 AMC 10 A, Problem #1—
2008 AMC 12 A, Problem #1—
"How many minutes did it take the machine to complete $\frac{1}{3}$ of the job?"

Solution

Answer (D): The machine worked for 2 hours and 40 minutes, or 160 minutes, to complete one third of the job, so the entire job will take $3 \cdot 160 = 480$ minutes, or 8 hours. Hence the doughnut machine will complete the job at 4:30 PM.

Difficulty: Easy
NCTM Standard: Measurement Standard: understand measurable attributes of objects and the units, systems, and processes of measurement.
Mathworld.com Classification: Number Theory > Arithmetic > General Arithmetic

A square is drawn inside a rectangle. The ratio of the width of the rectangle to a side of the square is 2:1. The ratio of the rectangle's length to its width is 2:1. What percent of the rectangle's area is inside the square?

(A) 12.5 **(B)** 25 **(C)** 50 **(D)** 75 **(E)** 87.5

2008 AMC 10 A, Problem #2—

"The ratio of length of the rectangle to a side of the square is 4:1 ."

Solution

Answer (A): Let x be the side length of the square. Then the area of the square is x^2. The rectangle has sides of length $2x$ and $4x$, and hence area $8x^2$. The fraction of the rectangle's area inside the square is $\frac{x^2}{8x^2} = \frac{1}{8}$ or 12.5%.

Difficulty: Medium
NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Geometry > Plane Geometry > Miscellaneous Plane Geometry > Area

For the positive integer n, let $<n>$ denote the sum of all the positive divisors of n with the exception of n itself. For example, $<4> = 1 + 2 = 3$ and $<12> = 1+2+3+4+6 = 16$. What is $<<<6>>>$?

(A) 6 **(B)** 12 **(C)** 24 **(D)** 32 **(E)** 36

2008 AMC 10 A, Problem #3—

"$<6> = 1 + 2 + 3 = 6$."

Solution

Answer (A): The positive divisors of 6, other than 6, are 1, 2, and 3, so $<6> = 1 + 2 + 3 = 6$. As a consequence, we also have $<<<6>>> = 6$. Note: A positive integer whose divisors other than itself add up to that positive integer is called a *perfect number*. The two smallest perfect numbers are 6 and 28.

Difficulty: Medium-easy
NCTM Standard: Number and Operations Standard: understand meanings of operations and how they relate to one another.
Mathworld.com Classification: Calculus and Analysis > Functions > Unary Operation

Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

(A) 2 **(B)** $\frac{5}{2}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** 4

2008 AMC 10 A, Problem #4—
2008 AMC 12 A, Problem #3—
"One banana is worth as much as $8 \cdot \frac{3}{20} = \frac{6}{5}$ oranges."

Solution

Answer (C): Note that $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas, which are worth as much as 8 oranges. So one banana is worth as much as $8 \cdot \frac{3}{20} = \frac{6}{5}$ oranges. Therefore $\frac{1}{2}$ of 5 bananas are worth as much as $\frac{5}{2} \cdot \frac{6}{5} = 3$ oranges.

Difficulty: Medium
NCTM Standard: Measurement Standard: make decisions about units and scales that are appropriate for problem situations involving measurement.
Mathworld.com Classification: Number Theory > Arithmetic > General Arithmetic

Which of the following is equal to the product

$$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdot\cdots\cdot\frac{4n+4}{4n}\cdot\cdots\cdot\frac{2008}{2004}?$$

(A) 251 **(B)** 502 **(C)** 1004 **(D)** 2008 **(E)** 4016

2008 AMC 10 A, Problem #5—
2008 AMC 12 A, Problem #4—
"Each denominator except the first can be canceled with the previous numerator."

Solution

Answer (B): Because each denominator except the first can be canceled with the previous numerator, the product is $\frac{2008}{4} = 502$.

Difficulty: Medium-hard
NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions

A triathlete competes in a triathlon in which the swimming, biking, and running segments are all of the same length. The triathlete swims at a rate of 3 kilometers per hour, bikes at a rate of 20 kilometers per hour, and runs at a rate of 10 kilometers per hour. Which of the following is closest to the triathlete's average speed, in kilometers per hour, for the entire race?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

2008 AMC 10 A, Problem #6— "Average Speed $= \frac{\text{Total Distance}}{\text{Total Time}}$."

Solution

Answer (D): Let x be the length of one segment, in kilometers. To complete the race, the triathlete takes

$$\frac{x}{3} + \frac{x}{20} + \frac{x}{10} = \frac{29}{60}x$$

hours to cover the distance of $3x$ kilometers. The average speed is therefore

$$\frac{3x}{\frac{29}{60}x} \approx 6 \text{ kilometers per hour.}$$

Difficulty: Medium-hard
NCTM Standard: Algebra Standard: use mathematical models to represent and understand quantitative relationships.
Mathworld.com Classification: Number Theory > Arithmetic > General Arithmetic

The fraction

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2}$$

simplifies to which of the following?

(A) 1 **(B)** $\frac{9}{4}$ **(C)** 3 **(D)** $\frac{9}{2}$ **(E)** 9

2008 AMC 10 A, Problem #7— "Factor 9^{2005} from each term."

Solution

Answer (E): First note that

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2} = \frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}}.$$

Factoring 9^{2005} from each of the terms on the right side produces

$$\frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}} = \frac{9^{2005} \cdot 9^3 - 9^{2005} \cdot 9^1}{9^{2005} \cdot 9^2 - 9^{2005} \cdot 1} = \frac{9^{2005}}{9^{2005}} \cdot \frac{9^3 - 9}{9^2 - 1} = 9 \cdot \frac{9^2 - 1}{9^2 - 1} = 9.$$

Difficulty: Hard
NCTM Standard: Number and Operations Standard: compute fluently and make reasonable estimates.
Mathworld.com Classification: Calculus and Analysis > Special Functions > Powers > Exponent

Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a \$90 rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves \$15 by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

(A) 750 **(B)** 900 **(C)** 1000 **(D)** 1050 **(E)** 1500

2008 AMC 10 A, Problem #8—
2008 AMC 12 A, Problem #6—
"Let x denote the sticker price, in dollars. Write the cost at each store in terms of x."

Solution

Answer (A): Let x denote the sticker price, in dollars. Heather pays $0.85x - 90$ dollars at store A and would have paid $0.75x$ dollars at store B. Thus the sticker price x satisfies $0.85x - 90 = 0.75x - 15$, so $x = 750$.

Difficulty: Medium
NCTM Standard: Algebra Standard: write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency–mentally or with paper and pencil in simple cases and using technology in all cases.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

is an integer. Which of the following statements must be true about x?

(A) It is negative. **(B)** It is even, but not necessarily a multiple of 3.

(C) It is a multiple of 3, but not necessarily even.

(D) It is a multiple of 6, but not necessarily a multiple of 12.

(E) It is a multiple of 12.

2008 AMC 10 A, Problem #9—
2008 AMC 12 A, Problem #5—
"$\frac{2x}{3} - \frac{x}{6} = \frac{x}{2}$."

Solution

Answer (B): Because

$$\frac{2x}{3} - \frac{x}{6} = \frac{x}{2}$$

is an integer, x must be even. The case $x = 4$ shows that x is not necessarily a multiple of 3 and that none of the other statements must be true.

Difficulty: Medium-hard
NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Number Theory > Integers

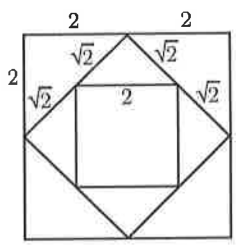
Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3. What is the area of S_3?

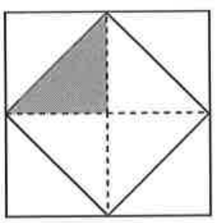
(A) $\frac{1}{2}$ **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2008 AMC 10 A, Problem #10—
"The sides of S_2 have length $\sqrt{2^2+2^2} = 2\sqrt{2}$."

Solution

Answer (E): The sides of S_1 have length 4, so by the Pythagorean Theorem the sides of S_2 have length $\sqrt{2^2+2^2} = 2\sqrt{2}$. By similar reasoning the sides of S_3 have length $\sqrt{(\sqrt{2})^2+(\sqrt{2})^2} = 2$. Thus the area of S_3 is $2^2 = 4$.





OR

Connect the midpoints of the opposite sides of S_1. This cuts S_1 into 4 congruent squares as shown. Each side of S_2 cuts one of these squares into two congruent triangles, one inside S_2 and one outside.
Thus the area of S_2 is half that of S_1. By similar reasoning, the area of S_3 is half that of S_2, and one fourth that of S_1.

Difficulty: Medium
NCTM Standard: Geometry Standard: apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Plane Geometry > Squares

While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

2008 AMC 10 A, Problem #11—
2008 AMC 12 A, Problem #7—
"It takes Steve 15 minutes to reach the shore."

Solution

Answer (D): At the rate of 4 miles per hour, Steve can row a mile in 15 minutes. During that time $15 \cdot 10 = 150$ gallons of water will enter the boat. LeRoy must bail $150 - 30 = 120$ gallons of water during that time. So he must bail at the rate of at least $\frac{120}{15} = 8$ gallons per minute.

OR

Steve must row for 15 minutes to reach the shore, so the amount of water in the boat can increase by at most $\frac{30}{15} = 2$ gallons per minute. Therefore LeRoy must bail out at least $10 - 2 = 8$ gallons per minute.

Difficulty: Medium
NCTM Standard: Algebra Standard: use mathematical models to represent and understand quantitative relationships.
Mathworld.com Classification: None

In a collection of red, blue, and green marbles, there are 25% more red marbles than blue marbles, and there are 60% more green marbles than red marbles. Suppose that there are r red marbles. What is the total number of marbles in the collection?

(A) $2.85r$ **(B)** $3r$ **(C)** $3.4r$ **(D)** $3.85r$ **(E)** $4.25r$

2008 AMC 10 A, Problem #12—

"Let b and g represent the number of blue and green marbles, respectively. Then $r = 1.25b$ and $g = 1.6r$."

Solution

Answer (C): Let b and g represent the number of blue and green marbles, respectively. Then $r = 1.25b$ and $g = 1.6r$. Thus the total number of red, blue, and green marbles is

$$r + b + g = r + \frac{r}{1.25} + 1.6r = r + 0.8r + 1.6r = 3.4r.$$

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

(A) $\left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1$ **(B)** $\left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$ **(C)** $\left(\frac{1}{5}+\frac{1}{7}\right)t=1$

(D) $\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$ **(E)** $(5+7)t=1$

2008 AMC 10 A, Problem #13—
2008 AMC 12 A, Problem #10—
"Working together, they can paint $\frac{1}{5}+\frac{1}{7}$ of the room in one hour."

Solution

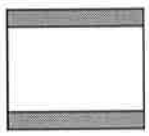
Answer (D): In one hour Doug can paint $\frac{1}{5}$ of the room, and Dave can paint $\frac{1}{7}$ of the room. Working together, they can paint $\frac{1}{5}+\frac{1}{7}$ of the room in one hour. It takes them t hours to do the job, but because they take an hour for lunch, they work for only $t-1$ hours. The fraction of the room that they paint in this time is

$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1),$$

which must be equal to 1. It may be checked that the solution, $t=\frac{47}{12}$, does not satisfy the equation in any of the other answer choices.

Difficulty: Medium-hard
NCTM Standard: Algebra Standard: use mathematical models to represent and understand quantitative relationships.
Mathworld.com Classification: None

Older television screens have an aspect ratio of 4:3. That is, the ratio of the width to the height is 4:3. The aspect ratio of many movies is not 4:3, so they are sometimes shown on a television screen by "letterboxing" — darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2:1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



(A) 2 **(B)** 2.25 **(C)** 2.5 **(D)** 2.7 **(E)** 3

2008 AMC 10 A, Problem #14—
2008 AMC 12 A, Problem #09—
"Apply the Pythagorean Theorem to find out height and width of the screen."

Solution

Answer (D): Let h and w be the height and width of the screen, respectively, in inches. By the Pythagorean Theorem, $h{:}w{:}27 = 3{:}4{:}5$, so

$$h = \frac{3}{5} \cdot 27 = 16.2 \quad \text{and} \quad w = \frac{4}{5} \cdot 27 = 21.6.$$

The height of the non-darkened portion of the screen is half the width, or 10.8 inches. Therefore the height of each darkened strip is

$$\frac{1}{2}(16.2 - 10.8) = 2.7 \quad \text{inches.}$$

OR

The screen has dimensions $4a \times 3a$ for some a. The portion of the screen not covered by the darkened strips has aspect ratio 2:1, so it has dimensions $4a \times 2a$. Thus the darkened strips each have height $\frac{a}{2}$. By the Pythagorean Theorem, the diagonal of the screen is $5a = 27$ inches. Hence the height of each darkened strip is $\frac{27}{10} = 2.7$ inches.

Difficulty: Medium-hard
NCTM Standard: Geometry Standard: use trigonometric relationships to determine lengths and angle measures.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Triangle Properties > Pythagorean Theorem

Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

(A) 120 **(B)** 130 **(C)** 140 **(D)** 150 **(E)** 160

2008 AMC 10 A, Problem #15—

"Set up equation to represent the relations with Ian's total time, h hours, and average speed, r miles."

Solution

Answer (D): Suppose that Ian drove for t hours at an average speed of r miles per hour. Then he covered a distance of rt miles. The number of miles Han covered by driving 5 miles per hour faster for 1 additional hour is

$$(r+5)(t+1) = rt + 5t + r + 5.$$

Since Han drove 70 miles more than Ian,

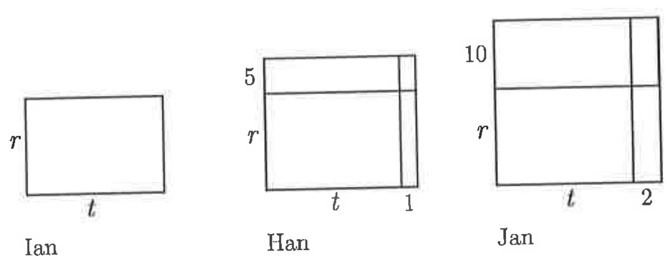
$$70 = (r+5)(t+1) - rt = 5t + r + 5, \quad \text{so} \quad 5t + r = 65.$$

The number of miles Jan drove more than Ian is consequently

$$(r+10)(t+2) - rt = 10t + 2r + 20 = 2(5t+r) + 20 = 2 \cdot 65 + 20 = 150.$$

OR

Represent the time traveled, average speed, and distance for Ian as length, width, and area, respectively, of a rectangle as shown. A similarly formed rectangle for Han would include an additional 1 unit of length and 5 units of width as compared to Ian's rectangle. Jan's rectangle would have an additional 2 units of length and 10 units of width as compared to Ian's rectangle.



Given that Han's distance exceeds that of Ian by 70 miles, and Jan's $10 \times t$ and $2 \times r$ rectangles are twice the size of Ian's $5 \times t$ and $1 \times r$ rectangles, respectively, it follows that Jan's distance exceeds that of Ian by

$$2(70-5) + 20 = 150 \text{ miles.}$$

Difficulty: Medium-hard
NCTM Standard: Algebra Standard: use symbolic algebra to represent and explain mathematical relationships.
Mathworld.com Classification: None

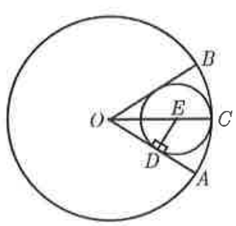
Points A and B lie on a circle centered at O, and $\angle AOB = 60^\circ$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

2008 AMC 10 A, Problem #16—
2008 AMC 12 A, Problem #13—
"$\angle AOC = \angle BOC = 30^\circ$."

Solution

Answer (B): Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90^\circ$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.



Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > 30-60-90 Triangle

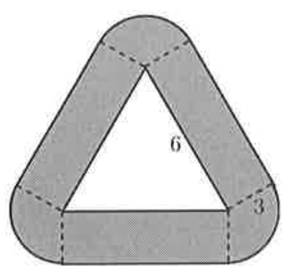
An equilateral triangle has side length 6. What is the area of the region containing all points that are outside the triangle and not more than 3 units from a point of the triangle?

(A) $36+24\sqrt{3}$ **(B)** $54+9\pi$ **(C)** $54+18\sqrt{3}+6\pi$ **(D)** $\left(2\sqrt{3}+3\right)^2\pi$ **(E)** $9\left(\sqrt{3}+1\right)^2\pi$

2008 AMC 10 A, Problem #17— "The region consists of three rectangles together with three $120°$ sectors of circles."

Solution

Answer (B): The region consists of three rectangles with length 6 and width 3 together with three $120°$ sectors of circles with radius 3.



The combined area of the three $120°$ sectors is the same as the area of a circle with radius 3, so the area of the region is

$$3 \cdot 6 \cdot 3 + \pi \cdot 3^2 = 54 + 9\pi.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Triangle

A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

(A) $\frac{57}{4}$ **(B)** $\frac{59}{4}$ **(C)** $\frac{61}{4}$ **(D)** $\frac{63}{4}$ **(E)** $\frac{65}{4}$

2008 AMC 10 A, Problem #18—

"Set up two equations from given information with three sides, and with the Pythagorean Theorem, we can generate another equation."

Solution

Answer (B): Let x be the length of the hypotenuse, and let y and z be the lengths of the legs. The given conditions imply that

$$y^2 + z^2 = x^2, \quad y + z = 32 - x, \quad \text{and} \quad yz = 40.$$

Thus

$$(32 - x)^2 = (y + z)^2 = y^2 + z^2 + 2yz = x^2 + 80,$$

from which $1024 - 64x = 80$, and $x = \frac{59}{4}$.
Note: Solving the system of equations yields leg lengths of

$$\frac{1}{8}(69 + \sqrt{2201}) \quad \text{and} \quad \frac{1}{8}(69 - \sqrt{2201}),$$

so a triangle satisfying the given conditions does in fact exist.

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze properties and determine attributes of two- and three-dimensional objects.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Triangle
Geometry > Plane Geometry > Triangles > Triangle Properties > Pythagorean Theorem

Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is rotated 90° clockwise about R, then rotated 90° clockwise about the point that S moved to after the first rotation. What is the length of the path traveled by point P?

(A) $\left(2\sqrt{3}+\sqrt{5}\right)\pi$ **(B)** 6π **(C)** $\left(3+\sqrt{10}\right)\pi$ **(D)** $\left(\sqrt{3}+2\sqrt{5}\right)\pi$

(E) $2\sqrt{10}\pi$

2008 AMC 10 A, Problem #19—
"Sketch the rotations."

Solution

Answer (C): Let P' and S' denote the positions of P and S, respectively, after the rotation about R, and let P'' denote the final position of P. In the rotation that moves P to position P', the point P rotates 90° on a circle with center R and radius $PR = \sqrt{2^2+6^2} = 2\sqrt{10}$. The length of the arc traced by P is $(1/4)\left(2\pi \cdot 2\sqrt{10}\right) = \pi\sqrt{10}$. Next, P' rotates to P'' through a 90° arc on a circle with center S' and radius $S'P' = 6$. The length of this arc is $\frac{1}{4}(2\pi \cdot 6) = 3\pi$. The total distance traveled by P is

$$\pi\sqrt{10} + 3\pi = \left(3+\sqrt{10}\right)\pi.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Transformations > Rotations > Rotation

Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

(A) 92 **(B)** 94 **(C)** 96 **(D)** 98 **(E)** 100

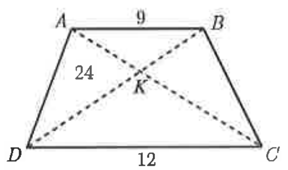
2008 AMC 10 A, Problem #20— "Note that $\triangle ABK$ is similar to $\triangle CDK$."

Solution

Answer (D): Note that $\triangle ABK$ is similar to $\triangle CDK$. Because $\triangle AKD$ and $\triangle KCD$ have collinear bases and share a vertex D,

$$\frac{\text{Area}(\triangle KCD)}{\text{Area}(\triangle AKD)} = \frac{KC}{AK} = \frac{CD}{AB} = \frac{4}{3},$$

so $\triangle KCD$ has area 32.
By a similar argument, $\triangle KAB$ has area 18. Finally, $\triangle BKC$ has the same area as $\triangle AKD$ since they are in the same proportion to each of the other two triangles. The total area is $24+32+18+24 = 98$.



OR

Let h denote the height of the trapezoid. Then

$$24 + \text{Area}(\triangle AKB) = \frac{9h}{2}.$$

Because $\triangle CKD$ is similar to $\triangle AKB$ with similarity ratio $\frac{12}{9} = \frac{4}{3}$,

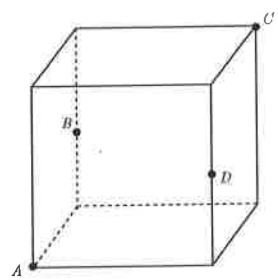
$$\text{Area}(\triangle CKD) = \frac{16}{9}\text{Area}(\triangle AKB), \quad \text{so} \quad 24 + \frac{16}{9}\text{Area}(\triangle AKB) = \frac{12h}{2}.$$

Solving the two equations simultaneously yields $h = \frac{28}{3}$. This implies that the area of the trapezoid is

$$\frac{1}{2} \cdot \frac{28}{3}(9 + 12) = 98.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Similar Triangles

A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$?



(A) $\frac{\sqrt{6}}{2}$ **(B)** $\frac{5}{4}$ **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$

(E) $\sqrt{3}$

2008 AMC 10 A, Problem #21— "All sides of $ABCD$ are of equal length."

Solution

Answer (A): All sides of $ABCD$ are of equal length, so $ABCD$ is a rhombus. Its diagonals have lengths $AC = \sqrt{3}$ and $BD = \sqrt{2}$, so its area is

$$\frac{1}{2}\sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}.$$

Difficulty: Hard

NCTM Standard: Geometry Standard: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals > Rhombus

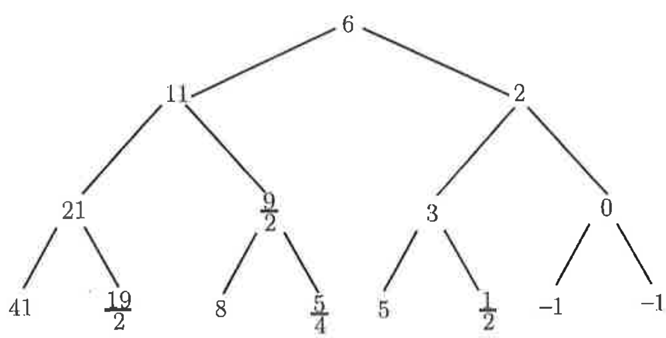
Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

2008 AMC 10 A, Problem #22—
"Draw a tree graph to see all possibilities at the fourth term."

Solution

Answer (D): The tree diagram below gives all possible sequences of four terms. In the diagram, each left branch from a number corresponds to a head, and each right branch to a tail.



Because the coin is fair, each of the eight possible outcomes in the bottom row of the diagram is equally likely. Five of those numbers are integers, so the required probability is $\frac{5}{8}$.

Difficulty: Medium-hard
NCTM Standard: Data Analysis and Probability Standard: understand and apply basic concepts of probability.
Mathworld.com Classification: Probability and Statistics > Probability

Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

2008 AMC 10 A, Problem #23—

"There are $\binom{5}{2} = 10$ ways to choose the two elements common to both subsets."

Solution

Answer (B): Let the two subsets be A and B. There are $\binom{5}{2} = 10$ ways to choose the two elements common to A and B. There are then $2^3 = 8$ ways to assign the remaining three elements to A or B, so there are 80 ordered pairs (A, B) that meet the required conditions. However, the ordered pairs (A, B) and (B, A) represent the same pair $\{A, B\}$ of subsets, so the conditions can be met in $\frac{80}{2} = 40$ ways.

Difficulty: Hard
NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Discrete Mathematics > Combinatorics > Permutations > Combination

Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2008 AMC 10 A, Problem #24— 2008 AMC 12 A, Problem #15— "The units digit of 2^n is always 2, 4, 8, or 6."

Solution

Answer (D): The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

Difficulty: Hard

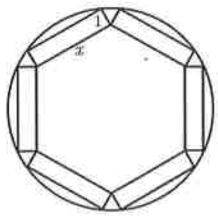
NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.

Mathworld.com Classification: Number Theory > Arithmetic > Number Bases > Digit

A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?

(A) $2\sqrt{5}-\sqrt{3}$ **(B)** 3 **(C)** $\dfrac{3\sqrt{7}-\sqrt{3}}{2}$ **(D)** $2\sqrt{3}$

(E) $\dfrac{5+2\sqrt{3}}{2}$

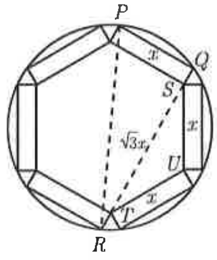


2008 AMC 10 A, Problem #25—
2008 AMC 12 A, Problem #22—
"Draw an auxiliary line from one mat corner to a diametrically opposite mat corner. Now make another auxiliary line to create a $30-60-90°$ triangle."

Solution

Answer (C): Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle $120°$. It follows that $ST = \sqrt{3}x$, so $QR = QS + ST + TR = \sqrt{3}x + 2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x+2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain

$$x = \frac{3\sqrt{7}-\sqrt{3}}{2}.$$



Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Triangle Properties > Pythagorean Theorem